A DOMINANT WEIGHT FOR $GL(n)$ IS $\lambda \in \mathbb{Z}^n$ SUCH THAT $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

IF $\lambda_n \geq 0$ THIS IS A PARTITION OF $|\lambda| := \sum \lambda_i$.

THEOREM (SCHUR, WEYL) GIVEN A DOMINANT WEIGHT THERE IS A UNIQUE IRR. OF $GL(n, \mathbb{C})$ WHOSE CHARACTER HAS THE FORM

$$\chi_\lambda \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} = z^\lambda + \text{OTHER LOWER TERMS}$$

$z^\lambda = z_1^{\lambda_1} \cdots z_n^{\lambda_n}$. IF λ IS A PARTITION

$$\chi_\lambda \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} = \Delta_\lambda (z_1, \dots, z_n)$$

SCHUR POLYNOMIAL.

"LOWER": PARTIAL ORDER ON $\mathbb{Z}^n$

$\lambda \geq \mu$ IF $\lambda_1 \geq \mu_1$, $\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2, \cdots$

$$\pi_\lambda \otimes \pi_\mu = \bigoplus_\nu c^\nu_{\lambda\mu}\, \pi_\nu .$$

THEOREM KASHIWARA, NAKASHIMA:

WITH λ AS ABOVE THERE IS A CRYSTAL $\mathcal{B}_\lambda$ WITH CHAR.

$$\chi_{\mathcal{B}_\lambda} = \Delta_\lambda(z_1, \dots, z_n)$$

$$\chi_{\mathcal{B}_\lambda} := \sum_{T \in \mathcal{B}_\lambda} z^{wt(T)} .$$

IF WE CHOOSE THE RIGHT CRYSTAL FOR ALL λ

$$\mathcal{B}_\lambda \otimes \mathcal{B}_\mu = \bigsqcup_\nu c^\nu_{\lambda\mu} \cdot \mathcal{B}_\nu$$

THE DECOMPOSITION OF TENSOR PRODUCTS OF REPS INTO IRR. MIRRORS THE DECOMPOSITION OF CRYSTALS INTO DISJOINT CRYSTALS.

IN HW

$$\overset{}{\mathcal{B}_{(1)}} \otimes \mathcal{B}_{(1,1)} = \overset{1}{\mathcal{B}_{(1,1,1)}} \sqcup \overset{8}{\mathcal{B}_{(2,1)}}$$

$\pi_{(1)} \otimes$

$$V \otimes V^* = \overset{1}{\mathbb{C}_{det}} \oplus \overset{8}{\text{"ADJOINT SQUARE"}}$$

STANDARD MODULE FOR $GL(3)$

THE COEFS $C^{\lambda}_{\mu\nu}$ ARE CALLED

LITTLEWOOD RICHARDSON COEFFICIENTS.

IF λ IS A PARTITION

$\mathcal{B}_\lambda$: ALL SSYT OF SHAPE λ IN $\{1, 2, \dots, n\}$.

HOW CAN I CONSTRUCT THIS ?

ANOTHER WAY THE REP'N THEORY AND CRYSTALS ARE COMPATIBLE.

IF $n = r + s$.

$$GL(r) \times GL(s) \hookrightarrow GL(n)$$

$$(g, h) \mapsto \left(\begin{array}{c|c} g & 0 \\ \hline 0 & h \end{array}\right)$$

WE CAN TAKE AN IRREP OF

$GL(n,\mathbb{C})$ AND RESTRICT IT TO $GL(r,\mathbb{C}) \times GL(n,\mathbb{C})$.

$$\Pi_\lambda^{GL_n}\Big|_{GL(r)\times GL(n)} = \bigoplus_{\mu,\nu} c_{\mu\nu}^{\lambda}\, \Pi_\mu^{GL_r} \otimes \Pi_\nu^{GL_n}$$

SURPRISINGLY COEFS $c_{\mu,\nu}^{\lambda}$ ARE ALSO LITTLEWOOD RICHARDSON COEFS.

FOR CRYSTALS YOU CAN GET THE CRYSTAL FOR $GL(r) \times GL(n)$ CRYSTAL BY ERASING SOME EDGES.

MORAL: CRYSTAL THEORY CLOSELY MIRRORS THE REP'N THEORY.

GL(4) CRYSTAL $\mathcal{B}_{(2)}$

$$
\begin{tikzcd}
\boxed{1\,1} \arrow[d, "1"] & & & \\
\boxed{1\,2} \arrow[r, "1"] \arrow[d, "2"] & \boxed{2\,2} \arrow[d, "2"] & & \\
\boxed{1\,3} \arrow[r, "1"] \arrow[d, "3"] & \boxed{2\,3} \arrow[r, "2"] \arrow[d, "3"] & \boxed{3\,3} \arrow[d, "3"] & \\
\boxed{1\,4} \arrow[r, "1"] & \boxed{2\,4} \arrow[r, "2"] & \boxed{3\,4} \arrow[r, "3"] & \boxed{4\,4}
\end{tikzcd}
$$

$V^2\, \mathbb{C}^4$ 10-DIM'L IRREP OF GL(4)

$\downarrow$

GL(2) × GL(2)

3 × 1 2 × 2

$\pi_{(2)} \otimes \pi_{(0)} \oplus \pi_{(1)} \otimes \pi_{(1)}$

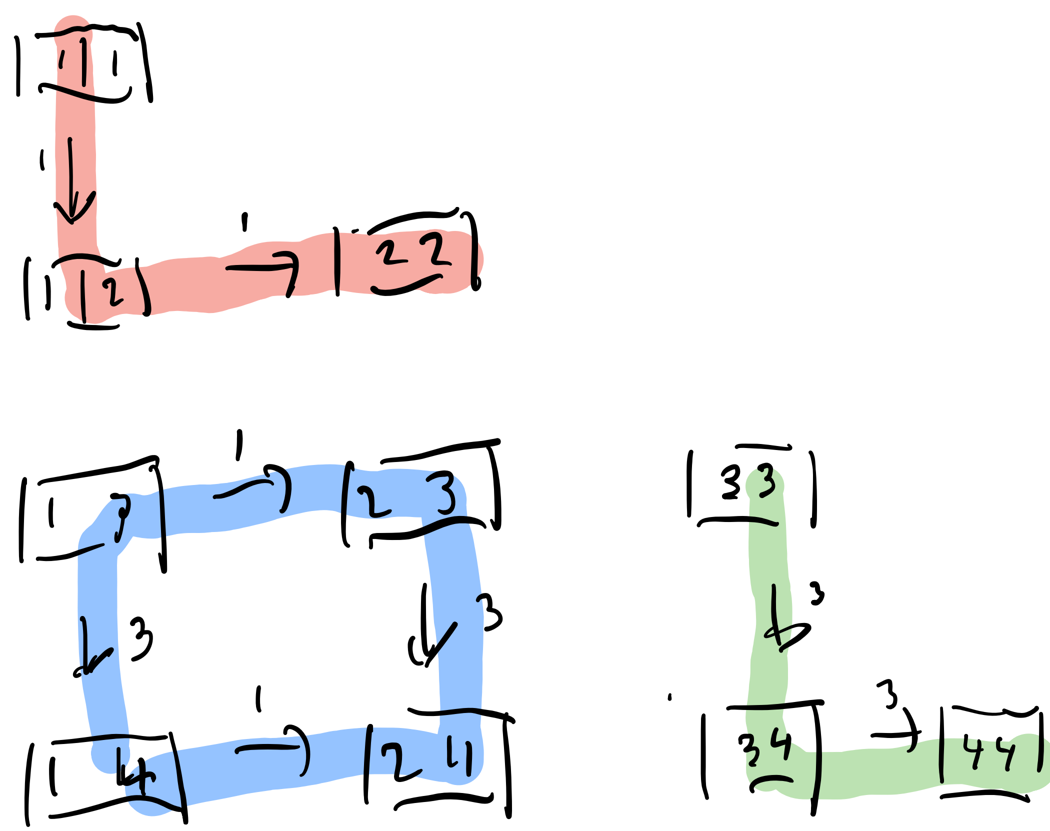
$\oplus\ \pi_{(0)} \otimes \pi_{(2)}$

$3+4+3=10$

BRANCHING RULE.

FOR CRYSTAL:

ERASE $\xrightarrow{2}$ EDGES.



$GL(2) \times GL(2) \hookrightarrow GL(4)$

f_1 $\quad$ f_3 $\quad$ f_1, f_2, f_3

$GL(r+s)$ CRYSTAL HAS $r+s-1$ SIMPLE ROOTS $\alpha_1, \dots, \alpha_{n-1}$

$n = r+s$.

$GL(r) \times GL(s)$ $r-1$ ROOTS IN $GL(r)$

$s-1$ ROOTS IN $GL(s)$

$\mathbb{Z}^2 \times \mathbb{Z}^2$ $(1,-1,0,0)$

$(1,-1)$ $(1,-1)$

$(0,0,1,-1)$

MISSING ROOT THAT IS A SIMPLE ROOT FOR $GL(4)$ BUT NOT $GL(2) \times GL(2)$ IS

α_2 $\cancel{(0,1,-1,0)}$.

THE CRYSTAL OPERATION IS DISCARD ALL EDGES LABELED 2.

ASSUME λ IS A PARTITION OF n

HOW TO CONSTRUCT

π_λ $\mathcal{B}_\lambda$.

PARALLEL CONSTRUCTIONS.

THEOREM: "SCHUR-WEYL DUALITY"

WE CAN FIND π_λ BY

TAKING $\bigotimes^n \mathbb{C}^n$

STANDARD MODULE

AND DECOMPOSING INTO IRREDUCIBLES.

MORE PRECISE:

$\bigotimes^k \mathbb{C}^n$ HAS

COMMUTING ACTIONS OF S_k, $GL(n, \mathbb{C})$

$$\bigotimes^k \mathbb{C}^n \cong \bigoplus_{\lambda \vdash k \ldots} \pi_\lambda^{S_k} \otimes \pi_\lambda^{GL(n,\mathbb{C})}.$$

$S_k \times GL(n)$

THIS MEANS MULTIPLICITY OF

$\pi_\lambda^{GL(n,\mathbb{C})}$ IN $\bigotimes^k \mathbb{C}^n$ IS

DIM $\pi_\lambda^{S_k}$.

S_3

		1	(123)	(12)
TRIVIAL	$\chi^{S_3}_{(3)}$	1	1	1
SIGN.	$\chi^{S_3}_{(111)}$	1	1	-1
2-DIM'L.	$\chi^{S_3}_{(21)}$	2	-1	0

So $\bigotimes^3 \mathbb{C}^n$ DECOMPOSE

INTO IRR. OF $GL(n)$

1 COPY OF $\vee^3 \mathbb{C}^n$

1 COPY OF $\Lambda^3 \mathbb{C}^n$

2 COPIES OF $\pi^{GL(n,\mathbb{C})}_{(2,1)}$.

$\|$

DIM $\pi_\lambda^{S_3}$.

SO OUR EXPECTATION

$$\mathcal{B}_{(1)} \otimes \mathcal{B}_{(1)} \otimes \mathcal{B}_{(1)} =$$

$$\mathcal{B}_{(3)} \sqcup \mathcal{B}_{(1,1,1)} \sqcup 2\,\mathcal{B}_{(2,1)}.$$

λ PARTITION OF k

THERE ARE MANY COPIES OF

$$\mathcal{B}_\lambda \quad \text{IN} \quad \underbrace{\mathcal{B} \otimes \cdots \otimes \mathcal{B}}_{k}$$

THERE IS A DISTINGUISHED ONE THAT IS EASY TO DESCRIBE.

CRYSTALS $\mathcal{B}_{(k)}$ AND $\mathcal{B}_{(\underbrace{1,\dots,1}_{k \text{ TIMES}})}$

CRYSTALS OF ROWS AND COLUMNS.

CRYSTAL $\mathcal{B}_{(k)} = \{ \boxed{i_1 \cdots i_k} \}$

$i_1 \leq i_2 \leq \cdots \leq i_k$

ROWS IS ISOMORPHIC TO

$$\{ \boxed{i_k} \otimes \cdots \otimes \boxed{i_1} \} \subseteq \underbrace{\mathcal{B} \otimes \cdots \otimes \mathcal{B}}_{k \text{ TIMES.}}$$

$\mathcal{B}^{GL(3)}_{(2)} =$

$$\{ \boxed{1} \otimes \boxed{1}, \boxed{2} \otimes \boxed{1}, \boxed{3} \otimes \boxed{1},$$
$$\boxed{2} \otimes \boxed{2}, \boxed{3} \otimes \boxed{2}, \boxed{3} \otimes \boxed{3} \}.$$

WE WANT TO CHECK THIS SUBSET

$\otimes^k \mathcal{B}$ $\qquad\qquad$ $\mathcal{B} = \mathcal{B}_{(1)}$

IS CLOSED UNDER e_i, f_i

COMBINATORIAL VERIFICATION. $\quad k = \boxed{2}$

$i \leq j$

$\boxed{j} \otimes \boxed{i} \qquad\qquad \boxed{i \mid j} \xrightarrow{f_i} \boxed{i+1 \mid j}$

UNLESS $i = j$

CRYSTALS OF COLUMNS.

$$\boxed{\begin{matrix} i_1 \\ \vdots \\ i_k \end{matrix}} \qquad i_1 < i_2 < \cdots < i_k$$

$$\rightsquigarrow \boxed{i_1} \otimes \cdots \otimes \boxed{i_k}$$

PRODUCES A SUBSET OF $\otimes^k \mathcal{B}$

THAT IS DISJOINT FROM CRYSTAL

OF ROWS.

GENERAL CASE:

$(\lambda_1, \dots, \lambda_r)$

A SSYT OF THIS SHAPE

CONSISTS OF ROWS.

$\lambda = (3,1)$

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \rightsquigarrow B_{(3)} \otimes B_{(1)}$$

$$\boxed{1\ 1\ 3} \otimes \boxed{2}$$

$$\rightsquigarrow B \otimes B \otimes B \otimes B$$

ALREADY EXPLAINED FOR CRYSTALS OF ROWS

$$\boxed{3} \otimes \boxed{1} \otimes \boxed{1} \otimes \boxed{2}$$

$$B_\lambda \rightsquigarrow B_{(\lambda_1)} \otimes B_{(\lambda_2)} \otimes \cdots$$

EACH ROW

$$\begin{matrix} r_1 \\ \vdots \\ r_n \end{matrix} \rightsquigarrow \boxed{r_1} \otimes \boxed{r_2} \otimes \cdots$$

$$\rightsquigarrow \otimes^{\lambda_1} B \oplus \otimes^{\lambda_2} B \oplus \cdots$$

REFERENCE SECTION 3.1 OF CRYSTAL BOOK.

NEXT TIME RSK: ROBINSON-SCHENSTED-KNUTH

FUNDAMENTAL TABLEAUX ALGORITHMS.

RSK IS A BIJECTION

WORDS $i_1, \dots, i_k$ $\quad 1 \leq i_j \leq n$

$\Updownarrow$

PAIRS OF TABLEAUX. T_1, T_2.

T_1 IS A SSYT

T_2 IS A STANDARD TABLEAU.

ENTRIES IN $\{1, 2, \dots, k\}$ $\quad \lambda \vdash k$

EVERY ROW, COLUMN STRICT

EACH ELT IN $\{1, \dots, k\}$ APPEARS ONCE.

FACT: IRR. REP $\Pi_\lambda^{S_k}$ HAS

DIMENSION = # OF STANDARD TABLEAUX OF SHAPE λ.